ON THE LINK BETWEEN FINITE DIFFERENCE AND DERIVATIVE OF POLYNOMIALS

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Abstract. The main aim of this paper to establish the relations between forward, backward and central finite (divided) differences (that is discrete analog of the derivative) and partial & ordinary high-order derivatives of the polynomials.

Keywords. finite difference, divided difference, high order finite difference, derivative, ode, pde, partial derivative, partial difference, power, power function, polynomial, monomial, power series, high order derivative, mathematics, differential calculus, differential equations

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1. Introduction

Let introduce the basic definition of finite difference. Finite difference is difference between function values with constant increment. There are three types of finite differences: forward, backward and central. Generally, the first order forward difference could be noted as: $\Delta_h f(x) = f(x+h) - f(x)$ backward, respectively, is $\nabla_h f(x) = f(x) - f(x-h)$ and central $\delta_h f(x) = f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)$, where $h = const$, (see [1], [2], [3]). When the increment is enough small, but

constant, we can say that finite difference divided by increment tends to derivative, but not equals. The error of this approximation could be counted next: $\frac{\Delta_h f(x)}{h} - f^{'}(x) = \mathcal{O}(h) \to 0$, where h - increment, such that, $h \to 0$. By means of induction as well right for backward difference. More exact approximation we have using central difference, that is: $\frac{\delta_h f(x)}{h} - f^{'}(x) = \mathcal{O}(h^2)$, note that function should be twice differentiable. The finite difference is the discrete analog of the derivative (see [4]), the main distinction is constant increment of the function's argument, while difference to be taken. Backward and forward differences are opposite each other. More generally, high order finite differences (forward, backward and central, respectively) could be denoted as (see [7]):

$$\Delta_h^k f(x) = \Delta^{k-1} f(x+h) - \Delta^{k-1} f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot f(x + (n-k)h) \tag{1.1}$$

$$\delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot f\left(x + \left(\frac{n}{2} - k\right) h\right) \tag{1.2}$$

$$\nabla_h^k f(x) = \nabla^{k-1} f(x) - \nabla^{k-1} f(x-h) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot f(x - kh) \tag{1.3}$$

Let describe the main properties of finite difference operator, they are next (see [5])

(1) Linearity rules $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$
$\delta(f(x) + g(x)) = \delta f(x) + \delta g(x)$
$\nabla(f(x) + g(x)) = \nabla f(x) + \nabla g(x)$
(2) $\Delta(C \cdot f(x)) = C \cdot \Delta f(x),\ \nabla(C \cdot f(x)) = C \cdot \nabla f(x)$,
$\delta(C \cdot f(x)) = C \cdot \delta f(x)$
(3) Constant rule $\Delta C = \nabla C = \delta C = 0$

Strictly speaking, divided difference (see [6]) with constant increment is discrete analog of derivative, when finite difference is discrete analog of function's differential. They are close related to each other. To show this, let define the divided difference.

Definition 1.4. Divided difference of fixed increment definition (forward, centaral, backward respectively)

$$f^{+}[x_i,\ x_j] := \frac{f(x_j) - f(x_i)}{x_j - x_i},\ j > i,\ \Delta x \geq 1$$

$$f^{-}[x_i,\ x_j] := \frac{f(x_i) - f(x_j)}{x_i - x_j},\ j < i,\ \nabla x \geq 1$$

$$f^{c}[x_i] := \frac{f(x_{i+m}) - f(x_{i-m})}{2m}$$

Hereby, divided diffrence could be represented from the finite difference, let be $j = i \pm \text{const}$, backward as $-$, respectively $+$ as forward and c as centered

$$f^{\pm}[x_i,\ x_j] \equiv \frac{\Delta f(x_j)}{\Delta x} \equiv \frac{\nabla f(x_i)}{\nabla x}$$

$$f_c[x_i] \equiv \frac{\delta f(x_{i\pm m})}{2m} \equiv \frac{\delta f(x_{i\pm m})}{\delta x}$$

The n-order

$$f^{\pm}[x_i,\ x_j]^n \equiv \frac{\Delta^n f(x_j)}{\Delta x^n} \equiv \frac{\nabla^n f(x_i)}{\nabla x^n}$$
$$f_c[x_i]^n \equiv \frac{\delta^n f(x_{i\pm m})}{(2m)^n} \equiv \frac{\delta^n f(x_{i\pm m})}{\delta x^n}$$

Each properties, which holds for finite differences holds for divided differences as well.

2. Definitions for discrete distribution

Let be variable $x_g : x_g = g \cdot C,\ C = x_{g+1} - x_g = \text{const},\ C \in \mathbb{R}_{>0} \to x_g \in \mathbb{R}_{>0},\ g \in \mathbb{Z}$. To define the finite difference of function of such argument, we take $C = h$ and rewrite forward, backward and central differences of some analytically defined function $f(x_i)$ next way: $\Delta f(x_{i+1}) = f(x_{i+1}) - f(x_i),\ \nabla f(x_{i-1}) = f(x_i) - f(x_{i-1}),\ \delta f(x_i) = f\left(x_{i+\frac{1}{2}}\right) - f\left(x_{i-\frac{1}{2}}\right)$. The n-th differences of such a function could be written as

$$\Delta^n f(x_{i+1}) = \Delta^{n-1} f(x_{i+1}) - \Delta^{n-1} f(x_i) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot f(x_{i+n-k}) \tag{2.1}$$

$$\delta^n f(x_i) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot f\left(x_{i+\frac{n}{2}-k}\right) \tag{2.2}$$

$$\nabla^n f(x_{i-1}) = \nabla^{n-1} f(x_i) - \nabla^{n-1} f(x_{i-1}) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot f(x_{i-n+k}) \tag{2.3}$$

Let be differences $\Delta f(x_{i+1}),\ \delta f(x_i),\ \nabla f(x_{i-1})$, such that $i \in \mathbb{Z}$ and differences is taken starting from point i, which divides the space $\mathbb{Z}$ into $\mathbb{Z} = \mathbb{Z}^- \cup \mathbb{Z}^+$ symmetrically (note that $+/-$ symbols mean the left and right sides of start point $i = 0$, i.e backward and forward direction), this way we have $(i+1) \in \mathbb{Z}^+,\ (i-1) \in \mathbb{Z}^-,\ i \neq 0 \in (\mathbb{Z}^+, \mathbb{Z}^-)$. Let derive some properties of that distribution:

1. $\max(\mathbb{Z}^-) = \min(\mathbb{Z}^+) = i$
2. Forward difference is taken starting from $\min(\mathbb{Z}^+)$, while backward from $\max(\mathbb{Z}^-)$
3. $\text{card}(\mathbb{Z}^+) = \text{card}(\mathbb{Z}^-)$, i.e $\sum_{k \in \mathbb{Z}^+} 1 = \sum_{k \in \mathbb{Z}^-} 1$
4. Maximal order of forward difference in which it is not equal to zero is $\max(\mathbb{Z}^+)$
5. Maximal order of backward difference in which it is not equal to zero is $\min(\mathbb{Z}^-)$
6. Maximal order of central difference in which it is not equal to zero is $\max(\mathbb{Z}^+)$
7. Forward and backward difference equal each other by absolute value, while to be taken from $i = 0$

Limitation 2.4. Note that most expression generated as case of $i = 0$, so the initial start point of each difference and inducted expressions are 0.

Definitions 2.5. Generalized definitions complete this section

1. $\mathbb{Z}^+ := \mathbb{N}_1$ - positive integers
2. $\mathbb{Z}^- := \{-1,\ -2,\ \dots,\ \min(\mathbb{Z}^-)\}$ - negative integers

(3) $\{f,\ f(x),\ f(x_i)\} := x^n$ - power function, value of power function in point i of difference table
(4) $i = 0$ - initial point of every differentiating process, $\delta f(x_0)$ exist only for operator of centered difference (as per limitation 2.4)
(5) $x_i := i \cdot \Delta x \equiv \nabla x \equiv (\delta x)/2 = \sum \Delta x$ - value of function's argument in point i of difference table
(6) $\Delta x \equiv \nabla x \equiv (\delta x)/2$ - function's argument differentials, constant values $\in \mathbb{R}_{>0}$
(7) $\Delta f(x_{i+1}),\ \nabla f(x_{i-1})$ - forward and backward finite differences in points $i+1$ and $i-1$ of difference table
(8) $\delta f(x_i)$ - centered finite difference in point i of difference table
(9) $\Delta^0 f \equiv \delta^0 f \equiv \nabla^0 f \equiv f$

3. Difference and derivative of power function

Since the n-order polynomial defined as summation of argument to power multiplied by coefficient, with higher power n, let describe a few properties of finite (divided) difference of power function.

Lemma 3.1. *For each power function with natural number as exponent holds the equality between forward, backward and central divided differences, and derivative with order respectively to exponent and equals to exponent under factorial sign multiplied by argument differential to power.*

Proof. Let be function $f(x) = x^n,\ n \in \mathbb{N}$. The derivative of power function, $f^{'}(x) = nx^{n-1}$, so k-th derivative $f^{(k)}(x) = n \cdot (n-1) \cdots (n-k+1) \cdot x^{n-k},\ n > k$. Using limit notation, we have: $\lim_{m \to n^-} f^{(m)}(x) = f^{(n)}(x) = n!$. Let rewrite expressions (2.1, 2.2, 2.3) according to definition $x_i = i \cdot \Delta x$, note that $\Delta x \equiv \nabla x \equiv \delta x/2$. By means of power function multiplication property $(i \cdot \Delta x)^n = i^n \cdot \Delta x^n$, we can rewrite the n-th finite difference equations (2.1, 2.2, 2.3) as follows

$$\Delta^m(x_{i+1}^n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot \big(i + m - k\big)^m \cdot \Delta x^m,\ m < n \in \mathbb{N} \tag{3.2}$$

Using limit notation on divided by Δx to power (3.2), we obtain

$$\lim_{m \to n^-} \frac{\Delta^m(x_{i+1}^n)}{\Delta x^m} = \lim_{m \to n^-} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot \big(i + m - k\big)^m \tag{3.3}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot \big(i + n - k\big)^{n-0} = n!$$

Similarly, going from (2.3), backward n-th difference equals:

$$\lim_{m \to n^-} \frac{\nabla^m(x_{i-1}^n)}{\nabla x^m} = \lim_{m \to n^-} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot \big(i - m + k\big)^m \tag{3.4}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot \big(i - n + k\big)^{n-0} = n!$$

And n-th central (2.2), respectively

$$\lim_{m\to n^-} \frac{\delta^m f(x_i)}{\delta x^m} = \lim_{m\to n^-} \sum_{k=0}^{m} \binom{m}{k}(-1)^k \cdot \left(i+\frac{m}{2}-k\right)^m \tag{3.5}$$

$$= \sum_{k=0}^{n} \binom{n}{k}(-1)^k \cdot \left(i+\frac{n}{2}-k\right)^{n-0} = n!$$

As we can see the next conformities hold

$$\lim_{\Delta x\to 0} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\Delta x\to C} \frac{\Delta^n f}{\Delta x^n} \equiv n! \tag{3.6}$$

$$\lim_{\delta x\to 0} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\delta x\to C} \frac{\delta^n f}{\delta x^n} \equiv n! \tag{3.7}$$

$$\lim_{\nabla x\to 0} \frac{\nabla^n f}{\nabla x^n} \equiv \lim_{\nabla x\to C} \frac{\nabla^n f}{\nabla x^n} \equiv n! \tag{3.8}$$

$$\lim_{\Delta x\to C} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\delta x\to C} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x\to C} \frac{\nabla^n f}{\nabla x^n} \qquad \forall (C \in \mathbb{R}^+) \tag{3.9}$$

In partial case when $C = 0$

$$\frac{d^n f}{dx^n} \equiv \lim_{\delta x\to 0} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x\to 0} \frac{\nabla^n f}{\nabla x^n} \tag{3.10}$$

As well holds

$$\frac{df}{dx}(x_0) = \left| \lim_{\nabla x\to 0} \frac{\nabla f}{\nabla x}(x_0) \right| \tag{3.11}$$

$$\frac{d^n f}{dx^n} \equiv \lim_{\Delta x\to C} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\delta x\to C} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x\to C} \frac{\nabla^n f}{\nabla x^n}, \quad \forall (C \in \mathbb{R}^+) \tag{3.12}$$

where $f = x^n$. And there is exist the continuous derivative and difference of order $k \leq n$ since $f \in C^n$ class of smoothness. Thus, from (3.6, 3.7, 3.8), we can conclude

$$\frac{d^n x^n}{dx^n} = \frac{\Delta^n (x_{i+1}^n)}{\Delta x^n} = \frac{\delta^n (x_i^n)}{\delta x^n} = \frac{\nabla^n (x_{i-1}^n)}{\nabla x^n} = n!, \ (\Delta x, \ \delta x, \ \nabla x) \not\to dx \tag{3.13}$$

This completes the proof. □

Definition 3.14. We introduce the difference equality operator $E(f)$, such that

$$E(f) \stackrel{\text{def}}{=} \left(\frac{\Delta^n f}{\Delta x^n} = \frac{\delta^n f}{\delta x^n} = \frac{\nabla^n f}{\nabla x^n} \right) \tag{3.15}$$

Property 3.16. *Let be central difference written as $\delta_m f(x_i) = f(x_{i+m}) - f(x_{i-m})$ the n-th central difference of n-th power is ${\delta^n}_m (x_i^n) = n! \cdot 2m \cdot \delta x^n$, where $\delta x = x_{i+1} - x_i = \text{const}$.*

Going from lemma (3.1), we have next properties

(1) $\Delta^k (x_{i+1}^k) = \text{const}, \ (i+1) \in \mathbb{Z}+ : \max(\mathbb{Z}+) > k \longrightarrow \Delta^k (x_{i+1}^k) \equiv \Delta^k (x_i^k)$
(2) $\nabla^k (x_{i-1}^k) = \text{const}, \ (i-1) \in \mathbb{Z}- : -\min(\mathbb{Z}-) \ll k \longrightarrow \nabla^k (x_{i-1}^k) \equiv \nabla^k (x_i^k)$
(3) $\delta^k (x_i^k) = \text{const}, \ i \in \mathbb{Z}+ : \max(\mathbb{Z}+) > k \longrightarrow \delta^k (x_i^k) \equiv \delta^k (x_{i+j}^k)$
(4) $\forall ([i+1] \in \mathbb{Z}+, \ [i-1] \in \mathbb{Z}-) : \Delta^{k+j} (x_{i+1}^k) = \nabla^{k+j} (x_{i-1}^k) = 0,$
$j > 1, \text{ since } \Delta C \equiv \delta C \equiv \nabla C \equiv 0$

(5) $\forall(f = x^n,\ n \in \mathbb{N},\ k \leq n) : \Delta^k f = (-1)^{n-1} \cdot \nabla^k f.$
(6) $\Delta f(x_{i+1}) = |\nabla f(x_{i-1})|$
(7) $\delta^2 f(x_0) = 2 \cdot (\delta x)^n,\ \forall(f(x_j) = x_j^n,\ n \bmod 2 = 0)$
(8) $\forall n \bmod 2 = 0 : \delta^{2j+1} f(x_0) = 0,\ j \in \mathbb{N}_0$ (see Appendix 1 for reference)
(9) $\forall n \bmod 2 = 1 : \delta^{2j} f(x_0) = 0,\ j \in \mathbb{N}_1$

Hereby, according to above properties, we can write the lemma (3.1) for enough large sets $\mathbb{Z}^+,\ \mathbb{Z}^-$ as

$$\frac{d^n x^n}{dx^n} = \frac{\Delta^n(x_i^n)}{\Delta x^n} = \frac{\delta^n(x_i^n)}{\delta x^n} = \frac{\nabla^n(x_i^n)}{\nabla x^n} = n! \tag{3.17}$$

Or

$$\left(\frac{d}{dx}\right)^n x^n = E(x^n) = n! \tag{3.18}$$

4. Difference of polynomials

Let be polynomial $P_n(x_g)$ defined as

$$P_n(x_g) = \sum_{i=0}^{n} a_i x_g^i \tag{4.1}$$

Finite differences of such kind polynomial, are $\Delta P_n(x_i) = P_n(x_{i+1}) - P_n(x_i)$, $\nabla P_n(x_{i-1}) = P_n(x_i) - P_n(x_{i-1})$, $\delta P_n(x_i) = P_n\left(x_{i+\frac{1}{2}}\right) - P_n\left(x_{i-\frac{1}{2}}\right)$. Such way, according to the properties (1, 2, 3) from section 1, high order finite differences of polynomials could be written as:

$$\begin{aligned}\Delta^k P_n(x_{i+1}) &= \Delta^k(a_0 \cdot x_{i+1}^0 + \cdots + a_n \cdot x_{i+1}^n) = \Delta^k(a_0 \cdot x_{i+1}^0) + \cdots + \Delta^k(a_n \cdot x_{i+1}^n) \\ &= a_0 \cdot \Delta^k(x_{i+1}^0) + \cdots + a_n \cdot \Delta^k(x_{i+1}^n) \end{aligned} \tag{4.2}$$

Backward difference, respectively, is

$$\begin{aligned}\nabla^k P_n(x_{i-1}) &= \nabla^k(a_0 \cdot x_{i-1}^0 + \cdots + a_n \cdot x_{i-1}^n) = \nabla^k(a_0 \cdot x_{i-1}^0) + \cdots + \nabla^k(a_n \cdot x_{i-1}^n) \\ &= a_0 \cdot \nabla^k(x_{i-1}^0) + \cdots + a_n \cdot \nabla^k(x_{i-1}^n) \end{aligned} \tag{4.3}$$

And central

$$\begin{aligned}\delta^k P_n(x_i) &= \delta^k(a_0 \cdot x_i^0 + \cdots + a_n \cdot x_i^n) = \delta^k(a_0 \cdot x_i^0) + \cdots + \delta^k(a_n \cdot x_i^n) \\ &= a_0 \cdot \delta^k(x_i^0) + \cdots + a_n \cdot \delta^k(x_i^n) \end{aligned} \tag{4.4}$$

Above expressions hold for each build natural n-order polynomial.

Lemma 4.5. $\forall([i+1] \in \mathbb{Z}^+,\ [i-1] \in \mathbb{Z}^-) : \Delta^{k+j}(x_{i+1}^k) \equiv \nabla^{k+j}(x_{i-1}^k) \equiv 0,\ j \geq 1$

Proof. According to lemma (3.1), the n-th difference of n-th power is constant, consequently, the constant rule (3) holds $\Delta C = \delta C = \nabla C = 0$. □

According to lemma (4.5) and properties (2, 3), taking the limits of (4.2, 4.3, 4.4), receive:

$$\begin{aligned}\Delta^{k \to n} P_n(x_{i+1}) &= \lim_{k \to n} \left\{\Delta^k(a_0 \cdot x_{i+1}^0) + \cdots + \Delta^k(a_n \cdot x_{i+1}^0)\right\} \\ &= \Delta^n(a_n \cdot x_{i+1}^n) = a_n \cdot \Delta^n(x_{i+1}^n) \end{aligned} \tag{4.6}$$

$$\delta^{k\to n} P_n(x_i) = \lim_{k\to n} \left\{\delta^k(a_0 \cdot x_i^0) + \cdots + \delta^k(a_n \cdot x_i^n)\right\} \tag{4.7}$$
$$= \delta^n(a_n \cdot x_i^n) = a_n \cdot \delta^n(x_i^n)$$

$$\nabla^{k\to n} P_n(x_{i-1}) = \lim_{k\to n} \left\{\nabla^k(a_0 \cdot x_{i-1}^0) + \cdots + \nabla^k(a_n \cdot x_{i-1}^n)\right\} \tag{4.8}$$
$$= \nabla^n(a_n \cdot x_{i-1}^n) = a_n \cdot \nabla^n(x_{i-1}^n)$$

Since the n-th difference of n-th power equals to $n!$, we have theorem.

Theorem 4.9. *Each n-order polynomial has the constant n-th finite (divided) difference and derivative, which equals each other and equal constant times $n!$, where n is natural.*

Proof. According to limits (4.6, 4.7, 4.8), we have $\Delta^n P_n(x_{i+1}) = a_n \cdot \Delta^n(x_{i+1}^n)$, $\nabla^n P_n(x_{i-1}) = a_n \cdot \nabla^n(x_{i-1}^n)$, $\delta^n P_n(x_i) = a_n \cdot \delta^n(x_i^n)$, going from lemma (3.1), the n-th difference of n-order polynomial equals to $k_n \cdot n!$, the properties (1, 2, 3, 4) proofs that for enough large sets $\mathbb{Z}^+$, $\mathbb{Z}^-$ we have $\Delta^n(x_{i+1}^n) \equiv \Delta^n(x_i^n)$, $\delta^n(x_i^n) \equiv \delta^n(x_{i+j}^n)$, $\nabla^n(x_{i-1}^n) \equiv \nabla^n(x_i^n)$, $\min(\mathbb{Z}^-) \le n \le \max(\mathbb{Z}^+)$. Therefore, we have equality

$$\frac{d^n P_n(x)}{dx^n} = \frac{\Delta^n P_n(x_i)}{(\Delta x)^n} = \frac{\delta^n P_n(x_i)}{(\delta x)^n} = \frac{\nabla^n P_n(x_i)}{(\nabla x)^n} = a_n \cdot n! \tag{4.10}$$

Or, by means of definition (3.14) one has

$$\left(\frac{d}{dx}\right)^n P_n(x) = E(P_n(x)) = a_n \cdot E(x^n) \tag{4.11}$$

□

Property 4.12. *Let be a plot of $\nabla^k x_i^n(k)$, $i \in \mathbb{Z}^-$ (see Appendix 1, second line for reference)*

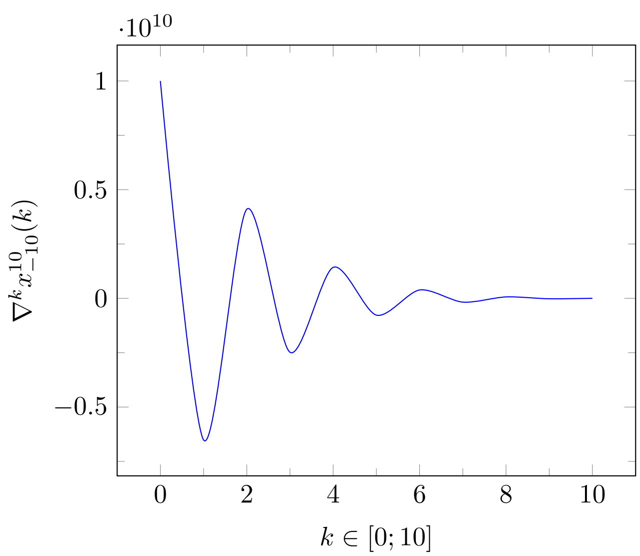

Figure 1. Plot of $\nabla^k x_i^n(k)$, $i \in \mathbb{Z}^-$

It's seen that each k-order backward difference (acc. to app 1) of power n, such that $n \geq k$ could be well interpolated by means of general Harmonic oscillator equation

$$x = A_0 e^{-\beta t} \sin(\omega t + \varphi_0) \tag{4.13}$$

Particularizing 4.13 to Figure 1, we get

$$\nabla^k x_i^n (j \leq k) = x^n e^{-\beta_i k} \sin(\omega_i k + \varphi_0) \tag{4.14}$$

In the points of local minimum and maximum of $\int x^n e^{-\beta k} \sin(\omega k + \varphi_0) dk$ we have $\nabla^k x^n,\ k \in [1;\ n] \subset \mathbb{N}_1$. By means of (5) we have relation with forward difference

$$\Delta^k x_i^n (k) = (-1)^{n-1} x^n e^{-\beta_i k} \sin(\omega_i k + \varphi_0) \tag{4.15}$$

Property 4.12 as well holds for polynomials.

5. Relation with Partial derivatives

Let be partial finite differences defined as

$$\Delta f(u_1,\ u_2, \ldots,\ u_n)_{u_1} := f(u_1 + h,\ u_2, \ldots,\ u_n) - f(u_1,\ u_2, \ldots,\ u_n) \tag{5.1}$$

$$\delta f(u_1,\ u_2, \ldots,\ u_n)_{u_1} := f(u_1 + h,\ u_2, \ldots,\ u_n) - f(u_1 - h,\ u_2, \ldots,\ u_n) \tag{5.2}$$

$$\nabla f(u_1,\ u_2, \ldots,\ u_n)_{u_1} := f(u_1,\ u_2, \ldots,\ u_n) - f(u_1 - h,\ u_2, \ldots,\ u_n) \tag{5.3}$$

By means of mathematical induction, going from Lemma (3.1), we have equality between n-th partial derivative and n-th partial difference, while be taken of polynomial defined function or power function.

Theorem 5.4. *For each n-th natural power of many variables the n-th partial divided differences and n-th partial derivatives equal each other.*

Proof. Let be function $Z = f(u_1,\ u_2, \ldots,\ u_n) = (u_1,\ u_2, \ldots,\ u_n)^n$, where dots mean the general relations, i.e multiplication and summation between variables. We denote the equality operator of partial difference as $E(F(u_1,\ u_2,\ \ldots,\ u_n))_{u_k}$, where u_k is variable of taken difference. On this basis

$$\frac{\partial^n Z}{\partial u_k^n} = \frac{\Delta^n Z_{u_k}}{\Delta u_k^n} = \frac{\delta^n Z_{u_k}}{\delta u_k^n} = \frac{\nabla^n Z_{u_k}}{\nabla u_k^n} = A \cdot n! \tag{5.5}$$

Or, using equality operator

$$\frac{\partial^n Z}{\partial u_k^n} = E(Z)_{u_k} = A \cdot n! \tag{5.6}$$

where A is free constant, depending of relations between variables and $0 \leq k \leq n$. ☐

Property 5.7. *Let be partial differences of the function $f(u_1,\ \cdots\ ,u_k) = u_1^n \pm u_2^n \pm\ \cdots\ \pm u_k^n,\ n \in \mathbb{N}$, $\Delta f(u_1,\ \cdots\ ,u_k)_M,\ \delta f(u_1,\ \cdots\ ,u_k)_M,\ \nabla f(u_1,\ \cdots\ ,u_k)_M$, where M - complete set of variables, i.e $M = \{u_i\}_i^k$ the n-th partial differences of each variables are*

$$\Delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3, \ldots, u_k} = \pm k \cdot n! \cdot (\Delta u_1)^n \cdots (\Delta u_k)^n \tag{5.8}$$

$$\delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3,\ \ldots, u_k} = \pm k \cdot n! \cdot (\delta u_1)^n \cdots (\delta u_k)^n \tag{5.9}$$

$$\nabla^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3,\ \ldots, u_k} = \pm k \cdot n! \cdot (\nabla u_1)^n \cdots (\nabla u_k)^n \tag{5.10}$$

$$\forall k \in \mathbb{Z} : \max(\mathbb{Z}^+) > n > \min(\mathbb{Z}^-),\ (\delta u_1) \equiv (\delta u_2) \equiv \ldots \equiv (\delta u_k),$$

$$(\nabla u_1) \equiv (\nabla u_2) \equiv \ldots \equiv (\nabla u_k),\ (\Delta u_1) \equiv (\Delta u_2) \equiv \ldots \equiv (\Delta u_k)$$

Otherwise

$$\Delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3,\ \ldots, u_k} = n! \cdot \sum_{i=1}^{k} (\Delta u_i)^n \tag{5.11}$$

$$\delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3,\ \ldots, u_k} = n! \cdot \sum_{i=1}^{k} (\delta u_i)^n \tag{5.12}$$

$$\nabla^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3,\ \ldots, u_k} = n! \cdot \sum_{i=1}^{k} (\nabla u_i)^n \tag{5.13}$$

Note that here the partial differences of non-single variable defined as

$$\Delta^n f(u_1^{,} \ldots, u_k)_M = \Delta^{n-1} f(u_1 + h, \ldots, u_k + h)_M - \Delta^{n-1} f(u_1, \ldots, u_k)_M$$
$$\delta^n f(u_1, \ldots, u_k)_M = \delta^{n-1} f(u_1 + h, \ldots, u_k + h)_M - \delta^{n-1} f(u_1 - h, \ldots, u_k - h)_M$$
$$\nabla^n f(u_1, \ldots, u_k)_M = \nabla^{n-1} f(u_1, \ldots, u_k)_M - \nabla^{n-1} f(u_1 - h, \ldots, u_k - h)_M$$

Moreover, the n-th partial difference taken over enough large set $\mathbb{Z}^+$ and $\forall i$: $\Delta x_i = 1$ has the next connection with single variable n-th derivative of n-th power

$$\Delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3, \ldots, u_k} = \sum_{i=1}^{k} \left(\frac{d}{du_i}\right)^n f(u_i) \tag{5.14}$$

With partial derivative we have relation

(5.15)
$$\Delta^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)_{u_1,\ u_2,\ u_3, \ldots, u_k} = \sum_{i=1}^{k} \left(\frac{\partial}{\partial u_i}\right)^n f(u_1,\ u_2,\ u_3,\ \ldots,\ u_k)$$

Multiplied (5.14) and (5.15) by coefficient, as defined, gives us relation with n-th partial polynomial.

Theorem 5.16. *For each non-single variable polynomial with order n holds the equality between $k \le n$-order partial differences and derivative.*

Proof. Let be non-single variable polynomial

$$\mathcal{P}_n(u_n) = \sum_{i=1}^{n} M_i \cdot u_i^i \tag{5.17}$$

Going from property (5.7), the k-th partial differences of one variable are

$$\Delta^k \mathcal{P}_n(u_n)_{u_k} = M_k \cdot k! \cdot (\Delta u_k)^k,\ \delta^k \mathcal{P}_n(u_n)_{u_k} = M_k \cdot k! \cdot (\delta u_k)^k, \tag{5.18}$$
$$\nabla^k \mathcal{P}_n(u_n)_{u_k} = M_k \cdot k! \cdot (\nabla u_k)^k$$

$0 \le k \le n$. The k-th partial derivative:

$$\frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = M_k \cdot k! \tag{5.19}$$

Hereby,

$$\frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = \frac{\Delta^k \mathcal{P}_n(u_n)_{u_k}}{\Delta u_k^k} = \frac{\delta^k \mathcal{P}_n(u_n)_{u_k}}{\delta u_k^k} = \frac{\nabla^k \mathcal{P}_n(u_n)_{u_k}}{\nabla u_k^k} \tag{5.20}$$

Also could be denoted as

$$\frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = E(\mathcal{P}_n(u_n))_{u_k} = M_k \cdot k!, \quad k \leq n \tag{5.21}$$

And completes the proof. □

6. Relations between finite differences

In this section are shown relations between central, backward and central finite differences, generally, they are

$$\delta_{\text{div}} f(x) := \frac{f(x+\Delta x) - f(x-\Delta x)}{2 \cdot \Delta x} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{f(x+\Delta x)}{\Delta x} - \frac{f(x-\nabla x)}{\nabla x}\right) \tag{6.1}$$

$$= \left| \begin{matrix} f(x+\Delta x) = \Delta f(x) + f(x) \\ f(x-\nabla x) = f(x) - \nabla f(x) \end{matrix} \right| = \frac{1}{2}\left(\frac{\Delta f(x) + f(x)}{\Delta x} - \frac{f(x) - \nabla f(x)}{\nabla x}\right)$$

$$= \frac{1}{2}\left(\frac{\Delta f(x) + \nabla f(x)}{\Delta x \equiv \nabla x}\right)$$

where "div" means divided, i.e $\delta_{\text{div}} f(x) := \delta f(x)/(2 \cdot \Delta x)$. Hereby,

$$2 \cdot \delta_{\text{div}} f(x) \cdot \Delta x = \Delta f(x) + \nabla f(x) \tag{6.2}$$

And so on. Let be $\Delta x \to 0$

$$\lim_{\Delta x \to 0} 2 \cdot \delta_{\text{div}} f(x) \cdot \Delta x = 2 \cdot df(x) \tag{6.3}$$

Or

$$2 \cdot \lim_{\Delta x \to 0} \delta_{\text{div}} f(x) = 2 \cdot \frac{df(x)}{dx} \longrightarrow \lim_{\Delta x \to 0} \delta_{\text{div}} f(x) = \frac{df(x)}{dx} \tag{6.4}$$

where $f(x)$ is power function, hence, the general relation between derivative and each kind finite difference is reached, as desired.

7. The error of approximation

The error of derivative approximation done by forward finite difference with respect to order $k \leq n$ could be calculated as follows

$$\left(\frac{\Delta}{\Delta x}\right)^k x^n - \left(\frac{d}{dx}\right)^k x^n = \mathcal{O}(x^{n-k}) \tag{7.1}$$

For n-order polynomial is

$$\left(\frac{\Delta}{\Delta x}\right)^k P_n(x) - \left(\frac{d}{dx}\right)^k P_n(x) = \mathcal{O}(x^{n-k}) \tag{7.2}$$

The partial, if $m \leq k$

$$\left(\frac{\Delta}{\Delta u_k}\right)^m Z - \left(\frac{\partial}{\partial u_k}\right)^m Z = \mathcal{O}(u_k^{k-m}) \tag{7.3}$$

Where $\mathcal{O}$ - Landau-Bachmann symbol (see [8], [9]).

8. Summary

In this section we summarize the obtained results in the previous chapters and establish the relationship between them. According to lemma (3.1), theorems (4.9), (5.4), (5.16) we have concluded

$$\frac{d^n x^n}{dx^n} = E(x^n) = n! \tag{8.1}$$

$$\frac{d^n P_n(x)}{dx^n} = E(P_n(x)) = a_n \cdot E(x^n) \tag{8.2}$$

$$\frac{\partial^n Z}{\partial u_k^n} = E(Z)_{u_k} = A \cdot n! \tag{8.3}$$

$$\frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = E(\mathcal{P}_n(u_n))_{u_k} = M_k \cdot k! \tag{8.4}$$

Generalizing these expressions, we can derive the general relations between ordinary, partial derivatives and finite (divided) differences

$$\underbrace{E(u^n) = E(P_n(u_g)) = E(Z)_{u_k} = E(\mathcal{P}_{n+j}(u_{n+j}))_{u_n}}_{Y} \tag{8.5}$$

$$\underbrace{\frac{d^n u^n}{du^n} = \frac{d^n P_n(u)}{du^n} = \frac{\partial^n Z}{\partial u_k^n} = \frac{\partial^n \mathcal{P}_{n+j}(u_{n+j})}{\partial u_n^n}}_{U}, \; j \geq 0 \tag{8.6}$$

$$\forall (A,\ M_n,\ a_n) = 1$$

I.e the equalities hold with precision to constant. Function Z defined as $Z = f(u_1,\ u_2, \ldots,\ u_n) = (u_1,\ u_2, \ldots,\ u_n)^n$. And finally

$$Y = U$$

with same limitations.

9. Conclusion

In this paper were established the equalities between ordinary and partial finite (divided) differences and derivatives of power function and polynomials, with order equal between each other.

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10. Appendix 1. Difference table up to tenth power

i	x_i	f	$(\Delta,\nabla)f/\delta^{10}f$	$(\Delta,\nabla)^2f/\delta^9f$	$(\Delta,\nabla)^3f/\delta^8f$	$(\Delta,\nabla)^4f/\delta^7f$	$(\Delta,\nabla)^5f/\delta^6f$	$(\Delta,\nabla)^6f/\delta^5f$	$(\Delta,\nabla)^7f/\delta^4f$	$(\Delta,\nabla)^8f/\delta^3f$	$(\Delta,\nabla)^9f/\delta^2f$	$(\Delta,\nabla)^{10}f/\delta f$
-10	-10	10000000000	-6513215599	4100173022	-2478397020	1425878520	-771309000	385363440	-172972800	66528000	-19958400	3628800
-9	-9	3486784401	-2413042577	1621776002	-1052518500	654569520	-385945560	212390640	-106444800	46569600	-16329600	**-8926258176**
-8	-8	1073741824	-791266575	569257502	-397948980	268623960	-173554920	105945840	-59875200	30240000	**7912982528**	-3204309152
-7	-7	282475249	-222009073	171308522	-129325020	95069040	-67609080	46070640	-29635200	**-6959124480**	2931599528	-1013275648
-6	-6	60466176	-50700551	41983502	-34255980	27459960	-21538440	16435440	**6063636480**	-2668596480	953858048	-272709624
-5	-5	9765625	-8717049	7727522	-6796020	5921520	-5103000	**-5225472000**	2415240960	-895488000	263003048	-59417600
-4	-4	1048576	-989527	931502	-874500	818520	**4443586560**	-2171473920	838164480	-253355520	58370048	-9706576
-3	-3	59049	-58025	57002	-55980	**-3715891200**	1937295360	-781885440	243767040	-57323520	9647528	-1047552
-2	-2	1024	-1023	1022	**3096576000**	-1703116800	727695360	-234178560	56279040	-9588480	1046528	-59048
-1	-1	1	-1	**-1857945600**	1703116800	-619315200	234178560	-54190080	9588480	-1044480	59048	-1024
0	0	0	**3715891200**	0	1238630400	0	108380160	0	2088960	0	2048	0
1	1	1	1	**1857945600**	1703116800	619315200	234178560	54190080	9588480	1044480	59048	1024
2	2	1024	1023	1022	**3096576000**	1703116800	727695360	234178560	56279040	9588480	1046528	59048
3	3	59049	58025	57002	55980	**3715891200**	1937295360	781885440	243767040	57323520	9647528	1047552
4	4	1048576	989527	931502	874500	818520	**4443586560**	2171473920	838164480	253355520	58370048	9706576
5	5	9765625	8717049	7727522	6796020	5921520	5103000	**5225472000**	2415240960	895488000	263003048	59417600
6	6	60466176	50700551	41983502	34255980	27459960	21538440	16435440	**6063636480**	2668596480	953858048	272709624
7	7	282475249	222009073	171308522	129325020	95069040	67609080	46070640	29635200	**6959124480**	2931599528	1013275648
8	8	1073741824	791266575	569257502	397948980	268623960	173554920	105945840	59875200	30240000	**7912982528**	3204309152
9	9	3486784401	2413042577	1621776002	1052518500	654569520	385945560	212390640	106444800	46569600	16329600	**8926258176**
10	10	10000000000	6513215599	4100173022	2478397020	1425878520	771309000	385363440	172972800	66528000	19958400	3628800

Note that central differences divided by **bold** typeset and kept in the middle of table. The table shows example in case $\max \mathbb{Z}^+ = 10,\ \min \mathbb{Z}^- = -10,\ \Delta x = 1,\ n = 10$